IA712: Mobile Robotics

Lecture 5: Kinematics

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What is Kinematics?

Definition:

Kinematics is the study of motion without considering the forces and torques that cause it. It describes the robot's position, velocity, and acceleration.

Key assumptions in this lecture:

- The robot is a rigid body.
- The wheels roll without slipping (the "no-slip" condition).
- Motion occurs on a 2D plane.

Objective:

To create a mathematical model that links the actuator speeds (wheel motors) to the velocity of the robot's body.

Forward vs. Inverse Kinematics

For a mobile robot, kinematics helps us answer two fundamental questions:

1. If my wheels turn at a certain speed, how is the robot's body moving through the world?
   Flow: From wheel speeds → to robot velocity.

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1. If my wheels turn at a certain speed, how is the robot's body moving through the world?
   Flow: From wheel speeds $\rightarrow$ to robot velocity.
   $\Longrightarrow$ Forward kinematics
2. To make the robot's body move in a desired direction at a certain speed, how fast do I need to turn my individual wheels?
   Flow: From robot velocity $\rightarrow$ to wheel speeds.
   $\Longrightarrow$ Inverse kinematics

Why is this crucial?

Forward kinematics

- It is the basis of odometry (estimating position).
- It allows the robot to know "where it is" based solely on its wheel movements.

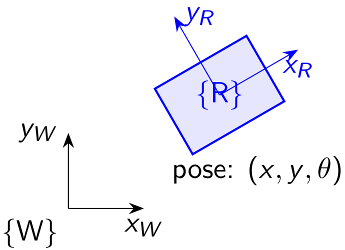
Inverse kinematics

- It is essential for any motor controller that takes velocity commands (like `/cmd_vel`) and translates them into actions.
- It allows the robot to execute a desired trajectory.

Frames of Reference

To describe motion, we must define our coordinate systems, or "frames".

- **Global frame (world frame):** A fixed, external reference frame.
  $\Longrightarrow$ Often called `map` or `odom` in ROS.
- **Local frame (robot frame):** A frame attached to the robot itself, typically at its center of rotation.
  $\Longrightarrow$ Often called `base_link` in ROS.



A robot's **pose** in the world is its pos. (x, y) and ori. θ w.r.t. the global frame.
$\Longrightarrow$ Kinematics allows us to calculate how this pose changes over time.

The Mathematics of Frames: Rotation

The relationship between the robot frame $\{R\}$ and the world frame $\{W\}$ is defined by a **translation** (x, y) and a **rotation** θ.

A point $P_R = (x_R, y_R)$ in the robot frame can be expressed in the world frame $P_W = (x_W, y_W)$ via a rotation.

2D rotation matrix:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The transformation is then: $P_W = R(\theta) \cdot P_R$.

Homogeneous Transformation Matrices

To combine rotation AND translation into a single matrix operation, we use homogeneous coordinates.

A point (x, y) becomes a vector $(x, y, 1)^T$.

Homogeneous transformation matrix T:

$$T = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R(\theta) & \mathbf{p} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

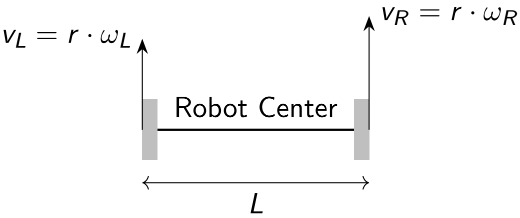
where $\mathbf{p} = (x, y)^T$ is the translation vector.

The full transformation of a point from the robot frame to the world frame is: $P_W = T \cdot P_R$.

The Differential Drive Model

Let's model the robot from Lecture 4:

- Two wheels separated by a distance L.
- Each wheel has a radius r.
- The left and right wheels rotate at angular velocities ω_L and ω_R.



Motion is controlled by varying the relative speed of the two wheels.

Goal: Given the wheel speeds (ω_L, ω_R), what are the linear velocity v and angular velocity ω of the robot's center?

Instantaneous Center of Curvature (ICC)

The motion of any rigid body in a plane can be described as a rotation around a single point, the Instantaneous Center of Curvature (ICC).

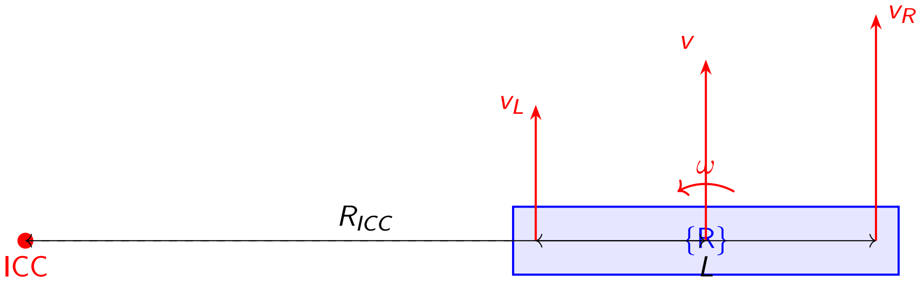
By varying ω_L and ω_R, the robot pivots around this point.

The distance R_{ICC} from the center of the robot to the ICC is given by:

$$\omega = \frac{v}{R_{ICC}} \iff R_{ICC} = \frac{v}{\omega}$$

where ω is the robot's angular velocity and v is its linear velocity.

Instantaneous Center of Curvature (ICC)



Deriving Forward Kinematics

The linear velocity of the robot, v, is the average of the two wheel velocities:

$$v = \frac{v_R + v_L}{2} = \frac{r(\omega_R + \omega_L)}{2}$$

The angular velocity of the robot, ω, is determined by the difference in wheel speeds, causing the robot to rotate around the ICC.

$$\omega = \frac{v_R - v_L}{L} = \frac{r(\omega_R - \omega_L)}{L}$$

Forward kinematic model:

These two equations allow us to map from the wheel velocity space (ω_L, ω_R) to the robot body velocity space (v, ω).

Note: The vector (v, ω) is exactly what a `geometry_msgs/msg/Twist` message in ROS represents!

Forward Kinematics: Special Cases

Moving in a straight line:

If $\omega_L = \omega_R$, then $\omega = 0$.
The robot moves straight ahead with velocity $v = r \cdot \omega_L$.
The ICC is at infinity.

Rotating in place:

If $\omega_L = -\omega_R$, then $v = 0$.
The robot spins about its center point with angular velocity $\omega = \frac{2r \cdot \omega_R}{L}$.
The ICC is at the robot's center.

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From Velocity to Pose: Odometry

The forward kinematics model gives us the robot's velocity (v, ω) in its **own frame** $\{R\}$.
To find the new pose (x', y', θ'), we must project this velocity into the **world frame** $\{W\}$ and integrate it.

Odometry equations:

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = \omega$$

Discrete update (Euler approximation):

$$x_{k+1} = x_k + v_k\cos(\theta_k)\Delta t$$
$$y_{k+1} = y_k + v_k\sin(\theta_k)\Delta t$$
$$\theta_{k+1} = \theta_k + \omega_k\Delta t$$

The Problem with Odometry: Drift

Odometry is a form of "dead reckoning".

Sources of error:

- Wheel slippage (uneven ground, rapid acceleration).
- Inaccurate wheel diameter, mechanical backlash.
- Time discretization.

Consequence:

- Errors accumulate without bound over time.
- The position estimate will inevitably drift.
- Odometry is reliable in the short term, but unusable alone in the long term.

Modeling Kinematic Errors & Uncertainty

Professionally, we don't just acknowledge errors, we model them.

Odometry errors are both **systematic** (e.g., one wheel is slightly larger) and **non-systematic** (e.g., random slip).

Error propagation:

The robot's velocity (v, ω) is not known perfectly. We represent this uncertainty with a covariance matrix:

$$\Sigma_v = \begin{pmatrix} \sigma_v^2 & \sigma_{v\omega} \\ \sigma_{v\omega} & \sigma_\omega^2 \end{pmatrix}$$

Σ_v in velocity must be propagated to the robot's pose (x, y, θ).

- Require computing the Jacobian of the motion model.
- The core mathematical basis of the **Extended Kalman Filter (EKF)** used in SLAM.

Deriving Inverse Kinematics

Goal: Given a desired linear velocity v and angular velocity ω, what are the required left and right wheel speeds (ω_L, ω_R)?

We simply need to rearrange our forward kinematics equations:

- $2v = v_R + v_L$
- $\omega L = v_R - v_L$

Solving for v_L and v_R gives us the wheel linear velocities:

$$v_R = v + \frac{\omega L}{2}$$

$$v_L = v - \frac{\omega L}{2}$$

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Final Step: To Motor Commands

We have the required linear velocities of the wheels (v_L, v_R). The final step is to convert these back to the angular velocities that a motor controller needs.

Inverse kinematic model:

$$\omega_R = \frac{v_R}{r} = \frac{1}{r}\left(v + \frac{\omega L}{2}\right)$$

$$\omega_L = \frac{v_L}{r} = \frac{1}{r}\left(v - \frac{\omega L}{2}\right)$$

Note: A robot's motor controller in ROS implements exactly these equations to turn a `/cmd_vel` message into motor speeds!

Beyond Differential Drive

Two other important models (cf. Lecture 4):

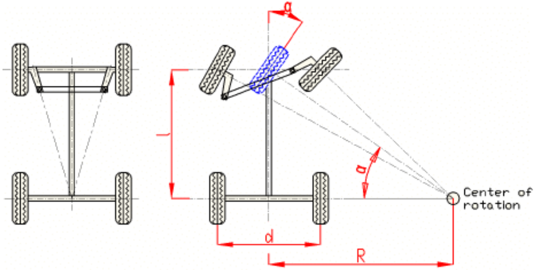
1. **Ackermann steering:** For car-like vehicles with steerable front wheels.
2. **Omnidirectional robots:** Capable of moving instantly in any direction.

Ackermann Kinematic Equations

The ICC lies on the axis of the rear wheel. The distance to the ICC depends on the steering angle α and the wheelbase L_{ack}: $R_{ICC} = \frac{L_{ack}}{\tan \alpha}$.

Forward Kinematics (bicycle model):

$$\dot{x} = v \cos \theta$$
$$\dot{y} = v \sin \theta$$
$$\dot{\theta} = \omega = \frac{v}{R_{ICC}} = \frac{v}{L_{ack}} \tan \alpha$$



$\Longrightarrow$ The command is (v, α), and the state is (x, y, θ).

Mecanum Wheel Kinematics

The kinematics are typically expressed in matrix form.

Inverse kinematics:

Calculates the speeds of the 4 wheels $(\omega_1, \ldots, \omega_4)$ from the body velocity command $(\dot{x}_R, \dot{y}_R, \omega_R)$ in the robot frame:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} = \frac{1}{r} \begin{pmatrix} 1 & -1 & -(L_x + L_y) \\ 1 & 1 & (L_x + L_y) \\ 1 & 1 & -(L_x + L_y) \\ 1 & -1 & (L_x + L_y) \end{pmatrix} \begin{pmatrix} \dot{x}_R \\ \dot{y}_R \\ \omega_R \end{pmatrix}$$

where L_x and L_y are the half-distances between the wheels.

Legged Robot Kinematics

- **Increased complexity:** Unlike a wheeled robot, a legged robot's "base" is not fixed relative to its actuators.
- **Approach:** Each leg is modeled as a **serial manipulator** (a kinematic chain).
- **Forward kinematics:** Given the angles of each joint (motor) in the leg, where is the foot tip relative to the robot's body?
- **Inverse kinematics:** To place the foot at a desired location, what angles must the joints have?

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The Role of Leg Kinematics in Locomotion

Why do we care about the foot position?

The Role of Leg Kinematics in Locomotion

Why do we care about the foot position?
$\Longrightarrow$ Stability!

- **Forward kinematics:** Determine the current locations of all feet on the ground.
  - These ground-contact points form a **Support Polygon**.
  - For the robot to be statically stable (i.e., not fall over when standing still), its **Center of Mass (CoM)** must be projected vertically inside this polygon.

- **Inverse kinematics:** Calculate the joint angles needed to place a swinging foot in a new location, creating a new support polygon and allowing the robot to move forward.

The Need for a Formal Description

So far, the kinematic parameters (r, L) are just numbers in our equations. But how does the entire ROS ecosystem know about the physical structure of our robot?

- How does a visualization tool like RViz know how to draw the robot?
- How does the system know where the LiDAR sensor is relative to the wheels?

Describing the Robot to ROS: URDF

URDF (Unified Robot Description Format):

An XML file format used to describe all the physical elements of a robot: its links and its joints.

- `<link>`: The rigid parts of the robot (chassis, wheel).
- `<joint>`: The kinematic relationship between two `links`.

Example of a wheel joint:

```
<joint name="left_wheel_joint" type="continuous">
<parent link="base_link"/>
<child link="left_wheel_link"/>
<origin xyz="0 0.15 -0.05" rpy="0 0 0"/>
<axis xyz="0 1 0"/>
</joint>
```

Managing Frames in ROS: tf2

Once the robot is described in URDF, ROS needs to know how the pose of each `<link>` changes over time.

tf2: The ROS transform library

- TF2 manages the relationship between all coordinate frames in the system.
- It maintains a tree of all transformations (e.g., `map` → `odom` → `base_link` → `lidar_link`).
- A node like `robot_state_publisher` uses the joint states and the URDF to continuously publish the tf2 transforms.
- This allows any other ROS node to ask "What is the LiDAR's position in the `map` frame?" at any time.

Questions?

Next: Practical Work 5 - tf2 & URDF